XXV Asian Pacific Mathematics Olympiad

Time allowed: 4 hours *Each problem if worth 7 points*

Problem 1. Let ABC be an acute triangle with altitudes AD, BE and CF, and let O be the center of its circumcircle. Show that the segments OA, OF, OB, OD, OC, OE dissect the triangle ABC into three pairs of triangles that have equal areas.

Problem 2. Determine all positive integers n for which $\dfrac{n^2+1}{[\sqrt{n}]^2+2}$ is an integer. Here $[r]$ denotes the greatest integer less than or equal to r.

Problem 3. For $2k$ real numbers $a_1, a_2, \ldots, a_k$, $b_1, b_2, \ldots, b_k$ define the sequence of numbers X_n by

$$X_n = \sum_{i=1}^{k} [a_i n + b_i] \quad (n = 1, 2, \ldots).$$

If the sequence X_n forms an arithmetic progression, show that $\sum_{i=1}^{k} a_i$ must be an integer. Here $[r]$ denotes the greatest integer less than or equal to r.

Problem 4. Let a and b be positive integers, and let A and B be finite sets of integers satisfying:

(i) A and B are disjoint;

(ii) if an integer i belongs either to A or to B, then $i + a$ belongs to A or $i - b$ belongs to B.

Prove that $a|A| = b|B|$. (Here $|X|$ denotes the number of elements in the set X.)

Problem 5. Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of AC such that PB and PD are tangent to ω. The tangent at C intersects PD at Q and the line AD at R. Let E be the second point of intersection between AQ and ω. Prove that B, E, R are collinear.